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New Classes of Exact Causal Viscous Cosmologies

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Two particular exact solutions of the gravitational field equations for a homogeneous flat Friedmann–Robertson–Walker universe filled with a causal bulk viscous fluid in the framework of the full Israel–Stewart–Hiscock theory are presented. The dynamics of the universe is entirely determined in the present models by its thermal behavior. The solutions of the field equations are expressed in an exact closed parametric form and correspond to an inflationary transition between a singular state and a Minkowskian space-time and a quasi-Minkowskian era and an inflationary state, connected by a noninflationary phase, respectively. The inflationary era of the first solution is associated with an increase in temperature and energy density (a heating period), but with a decrease of the comoving entropy. The evolutionary period described by the second solution leads during its noninflationary phase to a rapid decrease in the temperature, the energy density, and the comoving entropy of the corresponding universe.

1. INTRODUCTION

Dissipative bulk viscous-type thermodynamic processes are supposed to play a crucial role in the dynamics and evolution of the early universe. Over 30 years ago Misner (1966) suggested that the observed large-scale isotropy of the universe is due to the action of the neutrino viscosity which was effective when the universe was about 1 sec old. There are many processes capable of producing bulk viscous stresses in the early universe, like interaction between matter and radiation, quark and gluon plasma viscosity, and different components of dark matter (Chimento and Jakubi, 1996). Traditionally, for the description of these phenomena the theories of Eckart (1940) and Landau and Lifshitz (1987) were used. Because of the work of Israel (1976), Israel and Stewart (1976), and Hiscock and Lindblom (1989), it has become clear, however, that the Eckart-type theories suffer from serious

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drawbacks concerning causality and stability. Regardless of the choice of equation of state, all equilibrium states in these theories are unstable and in addition signals may be propagated through the fluid at velocities exceeding the speed of light (Israel, 1976; Israel and Stewart, 1976; Hiscock and Lindblom, 1989). These problems arise due to the first-order nature of the theory, i.e., it considers only first-order deviations from the equilibrium. The neglected second-order terms are necessary to prevent noncausal and unstable behavior.

A relativistic second-order theory was found by Israel (1976) and developed by Israel and Stewart (1976) into what is called 'transient' or 'extended' irreversible thermodynamics. However, Hiscock and Lindblom (1989) and Hiscock and Salmonson (1991) have shown that most versions of the causal second-order theories omit certain divergence terms. The truncated causal thermodynamics of bulk viscosity leads to pathological behavior in the late universe (Hiscock and Salmonson, 1991), while the solutions of the full causal theory are well behaved for all times. Therefore, the best currently available theory for analyzing dissipative processes in the universe is the full Israel–Stewart–Hiscock causal thermodynamics.

Due to the complicated nonlinear character of the evolution equations, very few exact cosmological solutions of the gravitational field equations are known in the framework of the full causal theory. Exact general solutions of the field equations have been obtained recently by Chimento and Jakubi (1997, 1998) for a homogeneous universe filled with a full causal viscous fluid source obeying the relation $\xi \sim \rho^{1/2}$, corresponding to a bulk viscosity coefficient ξ proportional to the Hubble factor H, and to two different choices of the state equations for pressure, bulk viscosity coefficient, temperature, and bulk relaxation time. Their solutions are expressed in an exact parametric form as two-parameter families of solutions.

In the present paper we present some other new classes of exact solutions of the gravitational field equations for a flat Friedmann–Robertson–Walker universe filled with an imperfect fluid having bulk viscosity under the framework of the full Israel–Stewart–Hiscock causal theory. They represent some particular solutions of the evolution equation for H corresponding to some fixed values of the constant physical parameters entering in the assumed equations of state of the cosmological fluid. The dynamics of the universe in these classes of models is entirely determined by its thermal behavior. The behavior of the energy density, the temperature, the bulk viscosity coefficient, the deceleration parameter, and the entropy is also analyzed.

The present paper is organized as follows. The field equations are written down in Section 2. A general solution of the field equations is presented in Section 3. In Section 4 we conclude and summarize our results.

2. THERMODYNAMICS, FIELD EQUATIONS, AND CONSEQUENCES

For a Robertson–Walker universe with a line element

$$ds^2 = dt^2 - a^2(t)\,(dx_1^2 + dx_2^2 + dx_3^2) \tag{1}$$

filled with a bulk viscous cosmological fluid the energy-momentum tensor is given by

$$T_i^k = (\rho + p + \Pi)\, u_i u^k - (p + \Pi)\delta_i^k \tag{2}$$

where ρ is the energy density, p is the thermodynamic pressure, Π is the bulk viscous pressure, and u_i is the four-velocity satisfying the condition $u_i u^i = 1$. We shall use units so that $8\pi G = 1$ and $c = 1$.

The gravitational field equations together with the continuity equation $T^k_{i;k} = 0$ imply

$$3\,H^2 = \rho \tag{3}$$

$$2\dot{H} + 3H^2 = -p - \Pi \tag{4}$$

$$\dot{\rho} + 3(\rho + p)H = -3\Pi H \tag{5}$$

The causal evolution equation for the bulk viscous pressure is given by (Maartens, 1995)

$$\tau\dot{\Pi} + \Pi = -3\xi H - \frac{1}{2}\,\tau\Pi\left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right) \tag{6}$$

where T is the temperature, ξ is the bulk viscosity coefficient, τ is the relaxation time, and we have denoted $H = \dot{a}/a$. Equation (6) arises as the simplest way (linear in Π) to satisfy the H theorem [i.e., for the entropy production to be nonnegative, $S^i_{;i} = \Pi^2/\xi T \geq 0$, where $S^i = sN^i - (\tau\Pi^2/2\xi T)u^i$ is the entropy flow vector, s is the entropy per particle, and $N^i = nu^i$ is the particle flow vector] (Israel and Stewart, 1976; Hiscock and Lindblom, 1989).

In order to close the system of equations (3)–(5) we have to give the equation of state for p and specify T, τ, and ξ. The equations of state for a homogeneous isotropic viscous fluid have been discussed by some authors. Hiscock and Salmonson (1991) used the equations of state arising from the Boltzmann equation to integrate numerically the gravitational field equations for a flat Friedmann–Robertson–Walker space-time. Lake (1982) considered a rather simplified equation of state given by the condition of the trace of the energy-momentum tensor being null. An analysis of the relativistic kinetic equation for some simple cases given by Murphy (1997), Belinskii and

Khalatnikov (1975), and Belinskii *et al.* (1979) shows that in the asymptotic regions of small and large values of energy density the viscosity coefficients can be approximated by power functions of the energy density with definite requirements on the exponents of these functions. For small values of the energy density it is reasonable to consider large exponents equal in the extreme case to one. For large ρ the power of the bulk viscosity coefficient should be considered smaller than (or equal to) 1/2. So, we shall assume the following simple phenomenological laws (Belinskii and Khalatnikov, 1975; Belinskii *et al.*, 1979; Zakari and Jou 1993; Maartens, 1995):

$$p = (\gamma - 1)\rho \tag{7}$$

$$\xi = \alpha\rho^{s} \tag{8}$$

$$T = \beta\rho^{r} \tag{9}$$

$$\tau = \frac{\xi}{\rho} = \alpha\rho^{s-1} \tag{10}$$

where $1 \leq \gamma \leq 2$, $\alpha \geq 0$, $\beta \geq 0$, $r \geq 0$, and $s > 0$ are constants. Equations (7)–(9) are standard in cosmological models, whereas equation (10) is a simple procedure to ensure that the speed of viscous pulses does not exceed the speed of light (Maartens, 1995).

The requirement that the entropy is a state function imposes in the present model the constraint (Chimento and Jakubi, 1997)

$$r = \frac{\gamma - 1}{\gamma} \tag{11}$$

so that $0 \leq r \leq 1/2$ for $1 \leq \gamma \leq 2$.

The growth of the total comoving entropy over a proper time interval (t_0, t) is given by (Maartens, 1995)

$$\Sigma(t) - \Sigma(t_0) = -\frac{3}{k}\int_{t_0}^{t} \frac{\Pi H a^3}{T}\, dt \tag{12}$$

where k is Boltzmann's constant.

The Israel–Stewart theory is derived under the assumption that the thermodynamic state of the fluid is close to equilibrium, that is, the nonequilibrium bulk viscous pressure should be small compared to the local equilibrium pressure (Zimdahl, 1996),

$$|\Pi| \ll p = (\gamma - 1)\rho \tag{13}$$

If this condition is violated, then one is effectively assuming that the linear theory holds also in the nonlinear regime far from equilibrium. For a fluid description of the matter, equation (13) ought to be satisfied (Zimdahl, 1996).

To see if a cosmological model inflates or not, it is convenient to introduce the deceleration parameter q,

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = \frac{\rho + 3p + 3\Pi}{2\rho} \tag{14}$$

The positive sign of the deceleration parameter corresponds to standard decelerating models, whereas the negative sign indicates inflation.

With these assumptions we obtain the following evolution equation for H (Maartens, 1995):

$$\ddot{H} + 3H\dot{H} + 3^{1-s}\alpha^{-1}H^{2-2s}\dot{H} - (1+r)H^{-1}\dot{H}^2 + \frac{9}{4}(\gamma - 2)H^3 + \frac{1}{2}3^{2-s}\alpha^{-1}\gamma H^{4-2s} = 0 \tag{15}$$

Equation (15) is transformed into the first order Abel-type differential equation for the unknown function $w(y)$ using the mathematical transformation $dy/d\eta = y^{(1+r)/2}/w$ and $s = 1/2$, the general solution of the Abel equation (consequently the field equations) is represented in an exact closed parametric form and corresponds to a transition between two Minkowskian space-time connected by an inflationary period (Mak and Harko, 1998a).

In our further investigation of equation (15) with $s = 1/2$ and relaxing the condition of equation (11), new classes of exact solutions of the field equations are generated from some particular solutions of the Abel equation leading to two classes of general solutions of the Einstein field equations corresponding to particular values of the parameters entering in the physical model. The solutions obtained are represented mathematically in an exact parametric form and are interpreted physically as describing cosmological particle production (Mak and Harko, 1998b).

By introducing a set of dimensionless variables h and θ by means of the transformations $H = (3^s\alpha/2)^{1/(1-2s)}h$, $t = 2/3(3^s\alpha/2)^{-1/(1-2s)}\theta$, $s \neq 1/2$, and using equation (11), we find that equation (15) takes the form

$$\frac{d^2h}{d\theta^2} + (2h + h^{2(1-s)})\frac{dh}{d\theta} - (1+r)h^{-1}\left(\frac{dh}{d\theta}\right)^2 + \frac{2r-1}{1-r}h^3 + \frac{1}{1-r}h^{2(2-s)} = 0 \tag{16}$$

Changing the variables according to

$$h = y^{1/(1-r)}, \qquad \eta = \int y^{1/(1-r)}\, d\theta \tag{17}$$

we find that equation (16) becomes

$$\frac{d^2y}{d\eta^2} + \left(2 + y^{(1-2s)/(1-r)}\right)\frac{dy}{d\eta} + (2r-1)y + y^{(1-2s)/(1-r)+1} = 0 \tag{18}$$

By the use of the mathematical substitution $u = dy/d\eta$, equation (18) is transformed into the following first-order differential equation for the unknown function u:

$$u\frac{du}{dy} + \left(2 + y^{(1-2s)/(1-r)}\right)u + (2r-1)y + y^{(1-2s)/(1-r)+1} = 0 \tag{19}$$

3. NEW CLASSES OF SOLUTIONS OF THE FIELD EQUATION

It is a matter of simple calculation to check that equation (19) has two particular solutions u_+ and u_- of the form

$$u_\pm = -\left[1 \pm \sqrt{2(1-r)}\right] y - \frac{1-r}{2-r-2s_\pm}\, y^{(1-2s_\pm)/(1-r)+1} \tag{20}$$

corresponding to particular values of $s = s_\pm$ related to r by means of the compatibility relation

$$s_\pm = \frac{1}{2} \pm 2\left(\frac{1-r}{2}\right)^{3/2} \tag{21}$$

The particular solutions of equation (20) together with equation (21) are the only solutions of this form satisfying the condition of the nonnegativity of s for all physically acceptable r, $s \geq 0$ for all $0 \leq r \leq 1/2$, and $s_+ \in [1/2 + 1/\sqrt{2}, 3/4)$ for $r \in [0, 1/2)$ for the first solution u_+ and $r \in [1 - 2^{-1/3}, 1/2)$ and $s_- \in [0, 1/4)$ for the second solution u_-.

By introducing a new parameter σ by means of the definition

$$\sigma = y^{-1/(1-r)} \tag{22}$$

and a dimensionless variable ε by means of the transformation $\rho = 3(3^s\alpha/2)^{2/(1-2s)}\,\varepsilon$, we can express the two particular solutions of the gravitational field equations for a Friedmann–Robertson–Walker isotropic space-time filled with a bulk viscous cosmological fluid in the framework of the full Israel–Stewart causal theory and corresponding to the particular values of $s_\pm$ given by equation (21) in the following exact closed parametric form:

$$\theta_{\pm} - \theta_{\pm 0} = (1 - r)(1 \mp \sqrt{2(1 - r)}) \int_{\sigma_0}^{\sigma} \frac{d\sigma}{\sigma^{\pm 4[(1-r)/2]^{3/2}} + 2r - 1} \tag{23}$$

$$a_{\pm} = a_{\pm 0} \left((2r - 1)\, \sigma^{\mp 4[(1-r)/2]^{3/2}} + 1\right)^{\mp\sqrt{2}(1 \mp \sqrt{2(1-r)})/[3(2r-1)\sqrt{1-r}]} \tag{24}$$

$$\varepsilon_{\pm} = \frac{1}{\sigma^2} \tag{25}$$

$$q_{\pm} = \frac{3}{2} \frac{\sigma^{\pm 4[(1-r)/2]^{3/2}} + 2r - 1}{(1 - r)(1 \mp \sqrt{2(1 - r)})} - 1 \tag{26}$$

$$\Sigma_{\pm}(\theta_{\pm}) - \Sigma_{\pm}(\theta_{\pm 0})$$

$$= S_{\pm 0} \int_{\sigma_0}^{\sigma} \frac{\sigma^{2r-3\mp 2(1-r)(1\mp\sqrt{2(1-r)})/[(2r-1)\sqrt{1-r}]}\left(1 - \{\sigma^{\pm 4[(1-r)/2]^{3/2}} + 2r - 1\}/[1 \mp \sqrt{2(1 - r)}]\right)}{(\sigma^{\pm 4[(1-r)/2]^{3/2}} + 2r - 1)^{1 \pm \sqrt{2}(1\mp\sqrt{2(1-r)})/[(2r-1)\sqrt{1-r}]}} d\sigma \tag{27}$$

where $a_{\pm 0}$ and $\theta_{\pm 0}$ constants of integration and we have denoted

$$S_{\pm 0} = \frac{2a_0^3[1 \mp \sqrt{2(1 - r)}]\, 3^{1-r}}{k\beta} \left(\frac{3^s\alpha}{2}\right)^{2(1-r)/(1-2s_{\pm})}$$

4. DISCUSSIONS AND FINAL REMARKS

The bulk viscous universe described by the solution corresponding to u_+ starts its evolution at $t = t_0$ from a nonsingular state characterized by finite values of the energy density, the bulk viscosity coefficient, and the temperature and the nonzero scale factor. The universe begins to expand and accelerated inflationary-type expansion occurs. During this period the deceleration parameter $q < 0$ for all values of r and of the parameter $\sigma > 0$. The inflationary behavior is associated with an increase of the energy density and of the temperature and so the universe experiences a heating period. This type of inflationary evolution has already been mentioned in the physical literature (Zimdahl, 1996). During the heating period the comoving entropy of the universe decreases, so in this model inflation is not associated with the production of a large amount of entropy. In the limit of large t, $t \to \infty$, $\sigma \to \infty$ and we have $\sigma^{4[(1-r)/2]^{3/2}} >> 2r - 1$. So, for large time the scale factor of the bulk viscous universe described by the first solution becomes a constant and the universe ends in a Minkowskian era. During the Minkowskian period there is a very slow increase in the energy density and the temperature.

The evolution of the universe described by the second solution u_- starts with a quasi-Minkowskian period corresponding to small times and values of σ satisfying the relation $(2r - 1)\sigma^{4[(1-r)/2]^{3/2}} << 1$. The universe begins to expand and the temperature and energy density decrease. The evolution of the universe is noninflationary ($q > 0$) for all times $t > 0$. The deceleration parameter is $q < 0$ for times $t > t_c$, where t_c is a critical value of the time (parameter σ), so the universe ends in an inflationary era. The expansion of the bulk viscous fluid-filled FRW universe described by the second solution is associated with a rapid decrease of the energy density, the temperature, and the comoving entropy.

The general condition for inflation $\ddot{a} > 0$ ($q < 0$) implies by equation (14) that $\rho + 3p + 3\Pi < 0$. This condition strongly violates equation (13), showing that for the first solution bulk viscous pressure is greater than the thermodynamic one, $|\Pi| > p$ for all times and consequently viscous fluid inflation is a far-from-equilibrium process (Maartens and Mendez, 1997). So, it is a matter for a future analysis and theoretical development to decide if the inflationary behavior characterizing the present exact solutions of the gravitational field equations for a FRW universe filled with a bulk viscous fluid given above can accurately describe a real physical period in the evolution of our universe. On the other hand, the second solution u_- leads to a noninflationary period in the evolution of the universe and satisfies $|\Pi| < p$ for a definite range of values s and r and for a finite period of time, leading to the possibility of correct physical description of a well-determined period in the evolution of our universe.

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